JOURNAL OF APPROXIMATION THEORY 4, 328–331 (1971)

A Best Two-Dimensional Space of Approximating Functions, II

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Communicated by Oved Shisha

Received February 2, 1970

Assume $\omega(x)$ is a given modulus of continuity, i.e., $\omega(x)$ is defined on $[0, \infty)$ and satisfies

(1) $\lim_{x\to 0+0} \omega(x) = \omega(0) = 0$.

(2) If $0 \leqslant x_1 < x_2$, then $0 \leqslant \omega(x_2) - \omega(x_1) \leqslant \omega(x_2 - x_1)$.

Let (M, ρ) be a given compact metric space and let $\Lambda_\omega(M, \rho)$ be the set of all real-valued $f(x)$ defined on M such that $|f(x) - f(y)| \leqslant \omega(\rho(x, y))$ for all $x, y \in M$.

In [1] we dealt with best 2-dimensional approximating spaces to $\Lambda_1(M, \rho)$ which is $\Lambda_\omega(M, \rho)$ for $\omega(x) = x$. In reviewing the methods used there, we found that they could be used to obtain similar results for $\Lambda_\omega(M, \rho)$ for any $\omega(x)$. However, we propose to show here how to get these results from those proved for $\Lambda_1(M, \rho)$. We shall also obtain a result which we were unable to prove by the last mentioned method, one which characterizes a best 2-dimensional approximating space to $\Lambda_\omega(M, \rho)$ for all $\omega(x)$.

Notation. (1) If $g_1(x)$ and $g_2(x)$ are any real functions on M, then $E_{\Lambda_\omega(M,\rho)}(g_1, g_2) = \sup_{f\in\Lambda_\omega(M,\rho)} \inf_{a,b} \|f - ag_1 - bg_2\|$ where $\|\ \|$ is the sup norm.

(2) $E_2(\Lambda_\omega(M, \rho)) = \inf_{g_1,g_2} E_{\Lambda_\omega(M,\rho)}(g_1, g_2)$.

(3) If $T \subseteq M$, then T' is its complement, $d_\rho(T) = \sup_{x,y\in T} \rho(x, y)$ is its ρ-diameter and $T(x)$ is its characteristic function, i.e.,

$$T(x) = \begin{cases} 1, & \text{if } x \in T, \\ 0, & \text{if } x \notin T. \end{cases}$$

Using this notation, the main theorems of [1] can be restated as follows:

THEOREM A. *There is a $T \subseteq M$ such that $E_{\Lambda_1(M,\rho)}(T, T') = E_2(\Lambda_1(M, \rho))$.*

* Partially supported by the United States Army Research Office (Durham).

THEOREM B. *Let $T \subseteq M$. Then $E_{\Lambda_1(M,\rho)}(T, T') = \frac{1}{2}\max(d_\rho(T), d_\rho(T'))$. In this article we generalize these results.*

LEMMA. *If $\omega(x) \not\equiv 0$, then $\omega\rho$ is a metric on M, where $\omega\rho(x, y) = \omega(\rho(x, y))$.*

Proof. (1) $\omega\rho(x, y) \geqslant 0$,

(2) $\omega\rho(x, y) = 0$ iff $x = y$,

(3) $\omega\rho(x, y) = \omega\rho(y, x)$,

(4) $\omega\rho(x, z) \leqslant \omega\rho(x, y) + \omega\rho(y, z)$.

Parts (1) and (3) are trivial. As to part (4), we have

$$\omega\rho(x, z) = \omega(\rho(x, z)) \leqslant \omega(\rho(x, y) + \rho(y, z)) \leqslant \omega(\rho(x, y)) + \omega(\rho(y, z))$$
$$= \omega\rho(x, y) + \omega\rho(y, z).$$

For part (2), if $x = y$, then trivially $\omega\rho(x, y) = 0$. If, however, $\omega\rho(x, y) = 0$ and $x \neq y$, then since ρ is a metric, $\rho(x, y) \neq 0$. Thus there is a $\delta > 0$ such that $\omega(\delta) = 0$. Since $\omega(x)$ is nondecreasing, $\omega(x) = 0$ for $x \in [0, \delta]$. For $x \in [\delta, 2\delta]$, $\omega(x) \leqslant 2\omega(x/2) = 0$. Thus, $\omega(x) = 0$ for $x \in [0, 2\delta]$. Continuing in this manner, we end up with $\omega(x) \equiv 0$, contradicting our hypothesis.

We note that if $\omega(x) \not\equiv 0$ and $T \subseteq M$, then $d_{\omega\rho}(T) = \omega(d_\rho(T))$.

We also note that if $\omega(x) \equiv 0$, then $\Lambda_\omega (M, \rho)$ is the set of all real-valued constant functions on M.

THEOREM 1. *There is a $T_\omega \subseteq M$ such that*

$$E_{\Lambda_\omega(M,\rho)}(T_\omega, T_\omega') = E_2(\Lambda_\omega (M, \rho)),$$

i.e., characteristic functions constitute a best approximating class.

Proof. If $\omega(x) \equiv 0$, then both sides of the equality are zero for any $T \subseteq M$ since $\Lambda_\omega (M, \rho)$ is one-dimensional (hence $E_2(\Lambda_\omega (M, \rho)) = 0$) and spanned by the constant function (hence $E_{\Lambda_\omega(M,\rho)}(T, T') = 0$).

If $\omega(x) \not\equiv 0$, then $\omega\rho$ is a metric on M and by its definition it is obvious that $f \in \Lambda_\omega (M, \rho)$ iff $f \in \Lambda_1 (M, \omega\rho)$. Thus $\Lambda_\omega (M, \rho) = \Lambda_1 (M, \omega\rho)$, $E_2(\Lambda_\omega (M, \rho)) = E_2(\Lambda_1 (M, \omega\rho))$ and $E_{\Lambda_\omega(M,\rho)}(g_1, g_2) = E_{\Lambda_1(M,\omega\rho)}(g_1, g_2)$ for any $g_1(x)$ and $g_2(x)$. From [1] we know that there is a $T_\omega \subseteq M$ such that

$$E_{\Lambda_1(M,\omega\rho)}(T_\omega, T_\omega') = E_2(\Lambda_1 (M, \omega\rho)).$$

Thus

$$E_{\Lambda_\omega(M,\rho)}(T_\omega, T_\omega') = E_2(\Lambda_\omega (M, \rho)).$$

THEOREM 2. *If* $T \subseteq M$, $E_{\Lambda_\omega(M,\rho)}(T, T') = \frac{1}{2}\omega(\max(d_\rho(T), d_\rho(T')))$.

Proof. Again, if $\omega(x) \equiv 0$, both sides are zero. If $\omega(x) \not\equiv 0$,

$$E_{\Lambda_\omega(M,\rho)}(T, T') = E_{\Lambda_1(M,\omega\rho)}(T, T').$$

However, by [1],

$$\begin{aligned} E_{\Lambda_1(M,\omega\rho)}(T, T') &= \tfrac{1}{2}\max(d_{\omega\rho}(T), d_{\omega\rho}(T')) \\ &= \tfrac{1}{2}\max(\omega(d_\rho(T)), \omega(d_\rho(T'))) \\ &= \tfrac{1}{2}\omega(\max(d_\rho(T), d_\rho(T'))). \end{aligned}$$

With these two theorems, we have generalized Theorems A and B. We have proved that characteristic functions form a best approximating space and we have also found out how to calculate the error. The next theorem relates this error to the one for $\omega(x) = x$.

THEOREM 3. $E_2(\Lambda_\omega(M, \rho)) = \frac{1}{2}\omega(2E_2(\Lambda_1(M, \rho)))$.

Proof. Since, for $T \subseteq M$, $\max(d_\rho(T), d_\rho(T')) = 2E_{\Lambda_1(M,\rho)}(T, T')$, Theorem 2 can be restated as $E_{\Lambda_\omega(M,\rho)}(T, T') = \frac{1}{2}\omega(2E_{\Lambda_1(M,\rho)}(T, T'))$. However, from Theorem 1 we have that

$$E_2(\Lambda_\omega(M, \rho)) = \inf_{T\subset M} E_{\Lambda_\omega(M,\rho)}(T, T').$$

Thus

$$\begin{aligned} E_2(\Lambda_\omega(M, \rho)) &= \inf_{T\subset M} (\tfrac{1}{2}\omega(2E_{\Lambda_1(M,\rho)}(T, T'))) \\ &= \tfrac{1}{2}\omega(\inf_{T\subset M} (2E_{\Lambda_1(M,\rho)}(T, T'))) \\ &= \tfrac{1}{2}\omega(2E_2(\Lambda_1(M, \rho))). \end{aligned}$$

Our next theorem states that, in Theorem 1, we can choose the $T \subseteq M$ independently of $\omega(x)$.

THEOREM 4. *There is a* $T \subseteq M$ *such that, for any modulus of continuity* $\omega(x)$,

$$E_{\Lambda_\omega(M,\rho)}(T, T') = E_2(\Lambda_\omega(M, \rho)).$$

Proof. Let $T \subseteq M$ be chosen such that $E_{\Lambda_1(M,\rho)}(T, T') = E_2(\Lambda_1(M, \rho))$.

Then

$$\begin{aligned} E_2(\Lambda_\omega(M, \rho)) &= \tfrac{1}{2}\omega(2E_2(\Lambda_1(M, \rho))) \\ &= \tfrac{1}{2}\omega(2E_{\Lambda_1(M,\rho)}(T, T')) \\ &= \tfrac{1}{2}\omega(\max(d_\rho(T), d_\rho(T')) \\ &= E_{\Lambda_\omega(M,\rho)}(T, T'). \end{aligned}$$

Reference

1. R. Feinerman, A Best Two-Dimensional Space of Approximating Functions, *J. Approximation Theory* **3** (1970), 50–58.